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ON THE FIRST FAILURE TIME OF DEPENDENT MULTICOMPONENT  
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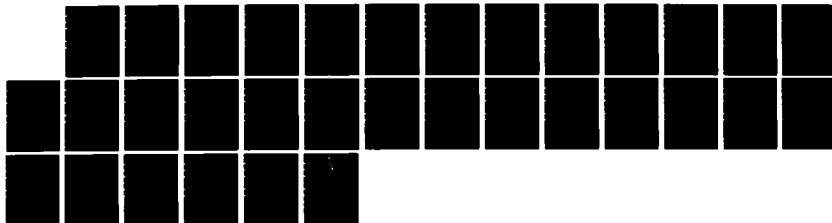
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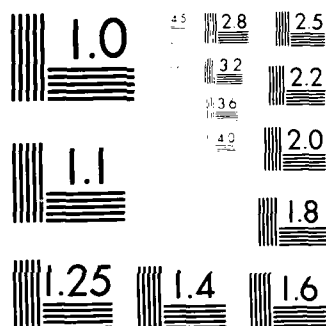
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ON THE FIRST FAILURE TIME OF  
DEPENDENT MULTICOMPONENT RELIABILITY  
SYSTEMS

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by

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### Abstract

In this paper are considered multicomponent reliability systems where component failure and repair completion rates depend on the state, ages and current repair durations of the other components. This is a generalization of a model of Ross (1984). Sufficient conditions on the sets of rates which imply stochastic ordering between first failure times of two such systems are found. Sufficient conditions on the rates which imply that the first failure time of such a system is new better than used (NBU) are given. Some results of Barlow and Proschan (1976), Chiang and Niu (1980) and Ross (1976) are obtained as special cases. A counterexample to an apparently stronger result of Miller (1979) is also given. Further results and a discussion are included.

Key words and phrases: Dependent maintained reliability system, coherent structures, repairable and nonrepairable components, performance process, NBU, IFR, stochastic ordering, uniformization by a Poisson process, stochastic monotonicity, NBU processes.

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## 1. Introduction.

The following model of dependent maintained reliability system (DMRS) with  $n$  components will be studied. At any time  $t$ , each component can be in one of two states: up (i.e., working) or down (i.e., failed and is in repair). The system state is also ~~either up or down~~ and it depends on the component states through a coherent structure function  $\phi$  (see, e.g., Barlow and Proschan (1975)). Let  $Z_i(t) = 0$  if component  $i$  is up at time  $t$ , let  $Z_i(t) = 1$  if component  $i$  is down at time  $t$  and let  $\underline{Z}(t) = (Z_1(t), \dots, Z_n(t))$ . Then the state of the system at time  $t$  is  $\phi(\underline{Z}(t))$ , that is,  $\phi(\underline{Z}(t)) = 0$  if the system is up and  $\phi(\underline{Z}(t)) = 1$  if the system is down.

We assume that the components are repairable, that is, as  $t$  varies,  $Z_i(t)$  alternates between intervals in which it is up and in which it is down,  $i = 1, 2, \dots, n$ . It is assumed that a repair [respectively, working] period starts immediately upon failure [respectively, repair completion] of a component.

We will study  $\{\phi(\underline{Z}(t)), t \geq 0\}$  through the multivariate process  $\underline{Z} = \{\underline{Z}(t), t \geq 0\}$  taking on values in  $\{0, 1\}^n$ . The process  $\underline{Z}$  will be called the performance process of the components of the DMRS. We assume that, with probability one,  $\underline{Z}$  has right continuous sample paths and that at any transition epoch of  $\underline{Z}$ , no more than one failure or one repair of a component can take place. That is, if for  $\underline{z}, \underline{m} \in \{0, 1\}^n$ ,  $\underline{z} - \underline{m} \equiv \sum_{i=1}^n (z_i - m_i)^2$ , then  $P\{\underline{Z}(t) - \underline{Z}(t-) \leq 1 \text{ for all } t\} = 1$ .

Note that if the  $n$  components are independently maintained and the components up and down periods are absolutely continuous then the process  $\underline{Z}$  of such an independently maintained reliability system (IMRS) satisfies the above condition. In this paper, however, the system is allowed to be

dependent in the sense that the durations of the up and down periods of a component can depend on the states of the other components.

Let  $W(\underline{Z}, t) \subset S \equiv \{1, \dots, n\}$  denote the set of the components which are working at time  $t$ , that is,  $W(\underline{Z}, t) \equiv \{i: Z_i(t) = 0\}$ . Clearly, for each  $t$ ,  $W(\underline{Z}, t)$  and  $\underline{Z}(t)$  determine each other. For any set  $w \subset S$ , denote the complement of  $w$  by  $\bar{w} \equiv S - w$ . Thus  $\bar{W}(\underline{Z}, t)$  is the set of the components which are under repair at time  $t$ .

For  $i \in W(\underline{Z}, t)$  [respectively,  $j \in \bar{W}(\underline{Z}, t)$ ] let  $A_i(\underline{Z}, t)$  [respectively,  $B_j(\underline{Z}, t)$ ] be the age of the current up [respectively, down] period of component  $i$  [respectively,  $j$ ] at time  $t$ . We will allow the instantaneous failure [respectively, repair completion] rate  $\lambda_i$  [respectively,  $\mu_j$ ] of component  $i$  [respectively,  $j$ ] to depend on the set of working components and on the ages of the current up and down periods of the other components. Such a generalization is needed when the working components share an overall load (see, e.g., Schechner (1984)) or when the repair facility has limited capacity. Thus, if at some time  $t$ ,  $W(\underline{Z}, t) = w \subset S$ ,  $A_i(\underline{Z}, t) = a_i > 0$ ,  $i \in w$  and  $B_j(\underline{Z}, t) = b_j > 0$ ,  $j \in \bar{w}$ , then for  $k \in w$ , we denote the instantaneous failure rate of component  $k$  by  $\lambda_k(w, \underline{a}_w, \underline{b}_{\bar{w}})$ , where  $\underline{a}_w \equiv (a_{i_1}, \dots, a_{i_m})$  when  $w = \{i_1, \dots, i_m\}$  and  $\underline{b}_{\bar{w}} = (b_{j_1}, \dots, b_{j_{n-m}})$  when  $\bar{w} = \{j_1, \dots, j_{n-m}\}$ . If  $w$  [respectively,  $\bar{w}$ ] is empty then  $\underline{a}_w$  [respectively,  $\underline{b}_{\bar{w}}$ ] is vacuous. Formally, for  $k \in w$ ,

$$(1.1) \quad \lambda_k(w, \underline{a}_w, \underline{b}_{\bar{w}}) \equiv \lim_{\Delta t \downarrow 0} \frac{1}{\Delta t} P\{Z_k(t + \Delta t) = 1 \mid W(\underline{Z}, t) = w, \underline{A}(\underline{Z}, t) = \underline{a}_w, \underline{B}(\underline{Z}, t) = \underline{b}_{\bar{w}}\}.$$

Similarly, define the instantaneous repair completion rate of component

$l \in \bar{w}$  by

$$(1.2) \quad \mu_k(w, \underline{a}_w, \underline{b}_w) \\ \equiv \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} P\{Z_k(t+\Delta t)=0 \mid W(\underline{Z}, t)=w, \underline{A}_w(\underline{Z}, t)=\underline{a}_w, \underline{B}_w(\underline{Z}, t)=\underline{b}_w\}.$$

Note that  $\lambda_k(w, \underline{a}_w, \underline{b}_w)$  and  $\mu_k(w, \underline{a}_w, \underline{b}_w)$  depend on  $t$  only through the current set of working components and the ages of the up and down periods of the components.

Note that in general the failure or the repair completion rate of component  $k$ , say, do not remain the same during periods between transitions of  $\underline{Z}$ . This is because  $a_i$ ,  $i \in w$ , and  $b_j$ ,  $j \in \bar{w}$ , increase linearly (with slope 1) during such periods.

In general the process  $\{\underline{Z}(t), t \geq 0\}$ , or equivalently  $\{W(\underline{Z}, t), t \geq 0\}$ , is not Markovian. However

$(W(\underline{Z}), \underline{A}_w(\underline{Z})(\underline{Z}), \underline{B}_w(\underline{Z})(\underline{Z})) \equiv \{(W(\underline{Z}, t), \underline{A}_w(\underline{Z}, t)(\underline{Z}, t), \underline{B}_w(\underline{Z}, t)(\underline{Z}, t)), t \geq 0\}$  is a Markov process. In the sequel we denote this process, when the initial state is  $W(\underline{Z}, 0) = w$ ,  $\underline{A}_w(\underline{Z}, 0) = \underline{a}_w$ ,  $\underline{B}_w(\underline{Z}, 0) = \underline{b}_w$ , by  $[(W(\underline{Z}), \underline{A}_w(\underline{Z})(\underline{Z}), \underline{B}_w(\underline{Z})(\underline{Z})) \mid w, \underline{a}_w, \underline{b}_w]$ . The process  $\underline{Z}$  which develops from this initial state will be denoted by  $[(\underline{Z}(t), t \geq 0) \mid w, \underline{a}_w, \underline{b}_w]$ .

When  $\lambda_i(w, \underline{a}_w, \underline{b}_w)$  and  $\mu_j(w, \underline{a}_w, \underline{b}_w)$  are independent of  $(w, \underline{a}_w, \underline{b}_w)$ , one obtains an IMRS with exponentially distributed up and down times.

Various aspects of the IMRS have been investigated by Barlow and Proschan (1975, 1976), Brown (1975, 1984), Chiang and Niu (1980), Keilson (1974, 1975), Miller (1979), Ross (1975, 1976) and Ross and Schechtman (1979). In particular Barlow and Proschan (1976), Theorem 2.7, and Chiang and Niu (1980), Theorem 3.5, provide an important characterization theorem for the distribution of the time to first system failure starting with all components new at time zero. One purpose of this paper is to extend this result to DMRS. Special cases of DMRS have been considered by Ross (1984) and Shanthikumar (1985).



In the reminder of this section we give some preliminaries on stochastic ordering and aging properties. In Section 2, we compare stochastically the performance processes of two different DMRS's. The results of Section 2 are used in Section 3 to obtain an aging property for some DMRS's. A counterexample for a related result (Theorem 3 of Miller, 1979) is given in Section 3.1. A discussion, further results and applications can be found in Section 4.

A random variable  $X$  is said to be stochastically smaller than [respectively, equal to] a random variable  $Y$  if  $P\{X > t\} \leq$  [respectively,  $=$ ]  $P\{Y > t\}$  for all  $t$ . We denote this relation by  $X \leq^t Y$  [respectively,  $X \equiv^t Y$ ]. A stochastic process  $\underline{X} = \{(X_1(t), \dots, X_n(t)), t \geq 0\}$  is said to be stochastically smaller than [respectively, equal to] a stochastic process  $\underline{Y} = \{(Y_1(t), \dots, Y_n(t)), t \geq 0\}$  if  $Eg(\underline{X}) \leq Eg(\underline{Y})$  [respectively,  $Eg(\underline{X}) = Eg(\underline{Y})$ ] for every nondecreasing functional  $g$  for which the expectations exist.

Let  $X$  be a random variable with distribution function  $F(\cdot)$  and survival function  $\bar{F}(\cdot) \equiv 1 - F(\cdot)$ . For each  $s$ , denote by  $[X-s|X>s]$  the random variable whose survival function is  $\bar{F}(s+)/\bar{F}(s)$ . The nonnegative random variable  $X$  is said to be new better than used (NBU) [respectively, new worse than used (NWU)] if  $X \leq^t [X-s|X>s]$  [respectively,  $X \geq^t [X-s|X>s]$ ] for all  $s \geq 0$ . It is said to have (or to be) increasing failure rate (IFR) [respectively, decreasing failure rate (DFR)] if  $[X-s|X>s] \geq^t [X-s'|X>s']$  [respectively,  $[X-s|X>s] \leq^t [X-s'|X>s']$ ] whenever  $s \leq s'$ . The latter condition is also equivalent to  $-\log \bar{F}(\cdot)$  being convex [respectively, concave].

## 2. Stochastic comparisons of two DMRS's.

Consider two DMRS's with performance processes

$\underline{X} = \{(X_1(t), \dots, X_n(t)), t \geq 0\}$  and  $\underline{Y} = \{(Y_1(t), \dots, Y_n(t)), t \geq 0\}$ . For  $w \in S \equiv \{1, \dots, n\}$ ,  $\underline{c}_w > 0$ ,  $\underline{d}_w \geq 0$ ,  $i \in \bar{w}$ ,  $j \in \bar{w}$ , let  $\alpha_i(w, \underline{c}_w, \underline{d}_w)$  and  $\beta_j(w, \underline{c}_w, \underline{d}_w)$  be, respectively, the failure and repair completion rates associated with  $\underline{X}$  defined analogously to (1.1) and (1.2). Similarly, let  $\gamma_i(w, \underline{e}_w, \underline{f}_w)$  and  $\delta_j(w, \underline{e}_w, \underline{f}_w)$  be the analogous rates associated with  $\underline{Y}$ .

The main result in this section shows that under some assumptions on the initial states of  $\underline{X}$  and  $\underline{Y}$  and on the rates  $\alpha_i, \beta_j, \gamma_i, \delta_j$ , it is possible to find processes  $\hat{\underline{X}} = \{(\hat{X}_1(t), \dots, \hat{X}_n(t)), t \geq 0\}$  and  $\hat{\underline{Y}} = \{(\hat{Y}_1(t), \dots, \hat{Y}_n(t)), t \geq 0\}$  defined on the same probability space, such that

$$(2.1) \quad \underline{X} \stackrel{st}{\leq} \hat{\underline{X}} \leq \hat{\underline{Y}} \stackrel{st}{\leq} \underline{Y},$$

where  $\hat{\underline{X}} \leq \hat{\underline{Y}}$  means  $P\{(\hat{X}_1(t), \dots, \hat{X}_n(t)) \leq (\hat{Y}_1(t), \dots, \hat{Y}_n(t)), t \geq 0\} = 1$ . It will follow then from (2.1) that  $\underline{X} \stackrel{st}{\leq} \underline{Y}$ .

In the statement of the next theorem and throughout the sequel, for  $w = \{i_1, \dots, i_m\}$ , the notation  $\underline{c}_w \leq \underline{e}_w$  means  $(c_{i_1}, \dots, c_{i_m}) \leq (e_{i_1}, \dots, e_{i_m})$ , i.e.  $c_{i_k} \leq e_{i_k}$  for every  $i_k \in w$ . The following assumption is needed for the uniformization procedure which is used in the proof of Theorem 2.2:

Assumption 2.1. For  $w \in v \in S$  and  $\underline{c}_w, \underline{d}_w, \underline{e}_v, \underline{f}_v$  denote

$$\begin{aligned}
\eta_i(w, v, \underline{c}_w, \underline{d}_w, \underline{e}_v, \underline{f}_v) &= \alpha_i(w, \underline{c}_w, \underline{d}_w) & \text{if } i \in w, \\
&= \delta_i(v, \underline{e}_v, \underline{f}_v) & \text{if } i \in \bar{v}, \\
&= \beta_i(w, \underline{c}_w, \underline{d}_w) \\
&\quad + \gamma_i(v, \underline{e}_v, \underline{f}_v) & \text{if } i \in \bar{w} \cap v.
\end{aligned}$$

Assume that

$$(2.2) \quad \tilde{\lambda} \equiv \sup \left\{ \sum_{i=1}^n \eta_i(w, v, \underline{c}_w, \underline{d}_w, \underline{e}_v, \underline{f}_v) : w \subset v \subset S, \underline{c}_w \leq \underline{e}_w, \underline{d}_w \geq \underline{f}_w \right\} < \infty$$

Theorem 2.2. Assume Assumption 2.1 and that for all choices of  $w, v \subset S$  and  $\underline{c}_w, \underline{d}_w, \underline{e}_v, \underline{f}_v$  such that  $w \subset v, \underline{c}_w \leq \underline{e}_w, \underline{d}_w \geq \underline{f}_w$ , we have

$$(2.3) \quad \alpha_i(w, \underline{c}_w, \underline{d}_w) \geq \gamma_i(v, \underline{e}_v, \underline{f}_v), \quad i \in w \subset v,$$

$$(2.4) \quad \beta_j(w, \underline{c}_w, \underline{d}_w) \leq \delta_j(v, \underline{e}_v, \underline{f}_v), \quad j \in \bar{v} \subset \bar{w}.$$

Then

$$[\underline{X}(t), t \geq 0 | w^1, \underline{c}_w^1, \underline{d}_w^1] \stackrel{st}{\approx} [\underline{Y}(t), t \geq 0 | v^1, \underline{e}_v^1, \underline{f}_v^1]$$

whenever

$$(2.5.i) \quad w^1 \subset v^1,$$

$$(2.5.ii) \quad \underline{c}_w^1 \leq \underline{e}_w^1,$$

$$(2.5.iii) \quad \underline{d}_w^1 \geq \underline{f}_w^1.$$

Proof. Let  $\lambda > \tilde{\lambda}, \lambda < \infty$ . such a  $\lambda$  exists by (2.2). Consider a Poisson process  $N = \{N(t), t \geq 0\}$  with intensity  $\lambda$ . Using  $N$ , we will construct two processes  $\hat{X} = \{\hat{X}(t), t \geq 0\}$  and  $\hat{Y} = \{\hat{Y}(t), t \geq 0\}$  which satisfy (2.1) and the desired result then follows.

Define  $\hat{\underline{X}}(0) \equiv (x_1(0), \dots, x_n(0))$  where  $x_i(0) = 0$  if  $i \in w^1$  and  $x_i(0) = 1$  if  $i \in \bar{w}^1$ . Similarly define  $\hat{\underline{Y}}(0) \equiv (y_1(0), \dots, y_n(0))$  where  $y_i(0) = 0$  if  $i \in v^1$  and  $y_i(0) = 1$  if  $i \in \bar{v}^1$ .

Let  $t_1$  be the time of the first epoch of  $N$ . For  $t \in [0, t_1)$  let  $\hat{\underline{X}}(t) = \underline{x}(0)$  and  $\hat{\underline{Y}}(t) = \underline{y}(0)$ . Thus, by (2.5.i),

$$(2.6) \quad \hat{\underline{X}}(t) \geq \hat{\underline{Y}}(t), \quad t \in [0, t_1).$$

Also, for  $t \in [0, t_1)$  let  $W(\hat{\underline{X}}, t) = w^1$ ,  $A_{\underline{W}}(\hat{\underline{X}}, t)(\hat{\underline{X}}, t) = \frac{c^1}{w^1} + t \underline{1}$ ,  $B_{\underline{W}}(\hat{\underline{X}}, t)(\hat{\underline{X}}, t) = \frac{d^1}{w^1} + t \underline{1}$ ,  $W(\hat{\underline{Y}}, t) = v^1$ ,  $A_{\underline{W}}(\hat{\underline{Y}}, t)(\hat{\underline{Y}}, t) = \frac{e^1}{v^1} + t \underline{1}$  and  $B_{\underline{W}}(\hat{\underline{Y}}, t)(\hat{\underline{Y}}, t) = \frac{f^1}{v^1} + t \underline{1}$ , where  $\underline{1}$  is a vector of ones. The dimension of  $\underline{1}$  may vary from one expression to another, but it is always possible to determine it from the formula in which  $\underline{1}$  appears.

Since  $\frac{c^1}{w^1} \leq \frac{e^1}{v^1}$  and  $\frac{d^1}{w^1} \geq \frac{f^1}{v^1}$  it follows that  $A_{\underline{W}}(\hat{\underline{X}}, t) \leq A_{\underline{W}}(\hat{\underline{Y}}, t)$  and  $B_{\underline{W}}(\hat{\underline{X}}, t) \geq B_{\underline{W}}(\hat{\underline{Y}}, t)$  for all  $t \in [0, t_1)$ .

For  $k > 0$  let  $t_k$  be the time of the  $k$ -th epoch of  $N$ . Assume, by induction, that  $\hat{\underline{X}}(t)$  and  $\hat{\underline{Y}}(t)$  have already been determined over the interval  $[0, t_k)$  and that, for  $t \in [0, t_k)$ ,  $\hat{\underline{X}}(t) \geq \hat{\underline{Y}}(t)$  [so that  $W(\hat{\underline{X}}, t) \subset W(\hat{\underline{Y}}, t)$ ] and that  $A_{\underline{W}}(\hat{\underline{X}}, t)(\hat{\underline{X}}, t) \leq A_{\underline{W}}(\hat{\underline{Y}}, t)(\hat{\underline{Y}}, t)$  and  $B_{\underline{W}}(\hat{\underline{X}}, t)(\hat{\underline{X}}, t) \geq B_{\underline{W}}(\hat{\underline{Y}}, t)(\hat{\underline{Y}}, t)$ . Denote the partial realization at time  $t_k^-$  by  $W(\hat{\underline{X}}, t_k^-) = w^k$ ,  $A_{\underline{W}}(\hat{\underline{X}}, t_k^-) = \frac{c^k}{w^k}$ ,  $B_{\underline{W}}(\hat{\underline{X}}, t_k^-) = \frac{d^k}{w^k}$ ,  $W(\hat{\underline{Y}}, t_k^-) = v^k$ ,  $A_{\underline{W}}(\hat{\underline{Y}}, t_k^-) = \frac{e^k}{v^k}$  and  $B_{\underline{W}}(\hat{\underline{Y}}, t_k^-) = \frac{f^k}{v^k}$ , so that by the induction hypothesis  $\frac{c^k}{w^k} \leq \frac{e^k}{v^k}$  and  $\frac{d^k}{w^k} \geq \frac{f^k}{v^k}$ .

Define  $\hat{\underline{X}}(t_k)$  and  $\hat{\underline{Y}}(t_k)$  as follows: At time  $t_k$  at most one component of  $\hat{\underline{X}}$  and at most one component of  $\hat{\underline{Y}}$  change as follows:

(a.1). With probability  $1 - \prod_{i=1}^n r_i(w^k, v^k, \frac{c^k}{w^k}, \frac{d^k}{w^k}, \frac{e^k}{v^k}, \frac{f^k}{v^k})$  the processes  $\hat{\underline{X}}$  and  $\hat{\underline{Y}}$  do not jump at time  $t_k$ . In this case

$$\begin{aligned}\hat{X}(t_k) &= \hat{X}(t_k^-), \quad W(\hat{X}, t_k) = w^k, \quad A_W(\hat{X}, t_k)(\hat{X}, t_k) = \frac{c^k}{w^k}, \\ B_W(\hat{X}, t_k)(\hat{X}, t_k) &= \frac{d^k}{w^k}, \quad \hat{Y}(t_k) = \hat{Y}(t_k^-), \quad W(\hat{Y}, t_k) = v^k, \\ A_W(\hat{Y}, t_k)(\hat{Y}, t_k) &= \frac{e^k}{v^k} \quad \text{and} \quad B_W(\hat{Y}, t_k)(\hat{Y}, t_k) = \frac{f^k}{v^k}.\end{aligned}$$

(a.2). With probability  $\lambda^{-1} \eta_r(w^k, v^k, \frac{c^k}{w^k}, \frac{d^k}{w^k}, \frac{e^k}{v^k}, \frac{f^k}{v^k})$  choose an index

$$r \in \{1, \dots, n\}, \quad \text{and for } i \neq r \quad \text{set } \hat{X}_i(t_k) = \hat{X}_i(t_k^-),$$

$$A_i(\hat{X}, t_k) = \frac{c_i^k}{w^k} \quad \text{when } i \in w^k, \quad B_i(\hat{X}, t_k) = \frac{d_i^k}{w^k} \quad \text{when } i \in \bar{w}^k,$$

$$\hat{Y}_i(t_k) = \hat{Y}_i(t_k^-), \quad A_i(\hat{Y}, t_k) = \frac{e_i^k}{v^k} \quad \text{when } i \in v^k, \quad \text{and}$$

$$B_i(\hat{Y}, t_k) = \frac{f_i^k}{v^k} \quad \text{when } i \in \bar{v}^k.$$

(b.1) If  $r \in w^k$ , then set  $\hat{X}_r(t_k) = 1$ ,  $W(\hat{X}, t_k) = w^k - \{r\}$ ,

$$B_r(\hat{X}, t_k) = 0. \quad \text{Also in this case, with probability}$$

$$\gamma_r(v^k, \frac{e^k}{v^k}, \frac{f^k}{v^k}) / \alpha_r(w^k, \frac{c^k}{w^k}, \frac{d^k}{w^k}) \quad [\text{which is } \leq 1 \text{ by (2.3)}] \text{ set}$$

$$\hat{Y}_r(t_k) = 1, \quad W(\hat{Y}, t_k) = v^k - \{r\}, \quad B_r(\hat{Y}, t_k) = 0, \quad \text{and with}$$

$$\text{probability } 1 - \gamma_r(v^k, \frac{e^k}{v^k}, \frac{f^k}{v^k}) / \alpha_r(w^k, \frac{c^k}{w^k}, \frac{d^k}{w^k}) \quad \text{set } \hat{Y}_r(t_k) = 0,$$

$$W(\hat{Y}, t_k) = v^k \quad \text{and} \quad A_r(\hat{Y}, t_k) = \frac{e_r^k}{v^k}.$$

Intuitively, under (b.1) one sees that component  $r$  has failed at time  $t_k$  in the realization of  $\hat{X}$ . As for  $\hat{Y}$ , component  $r$  either fails or does not fail. In either case it is seen that

$$(2.7) \quad \hat{X}(t_k) \geq \hat{Y}(t_k),$$

$$(2.8) \quad A_W(\hat{X}, t_k)(\hat{X}, t_k) \leq A_W(\hat{X}, t_k)(\hat{Y}, t_k),$$

$$(2.9) \quad B_W(\hat{Y}, t_k)(\hat{X}, t_k) \geq B_W(\hat{Y}, t_k)(\hat{Y}, t_k).$$

(b.2). If  $r \in \bar{v}^k$  then set  $\hat{Y}_r(t_k) = 0$ ,  $\bar{W}(\hat{Y}, t_k) = \bar{v}^k - \{r\}$ ,

$$A_r(\hat{Y}, t_k) = 0. \quad \text{Also in this case, with probability}$$

$$\beta_r(w^k, \frac{c^k}{w^k}, \frac{d^k}{w^k}) / \beta_r(v^k, \frac{e^k}{v^k}, \frac{f^k}{v^k}) \quad [\text{which is } \leq 1 \text{ by (2.4)}] \text{ set}$$

$$\hat{X}_r(t_k) = 0, \quad \bar{W}(\hat{X}, t_k) = \bar{w}^k - \{r\}, \quad A_r(\hat{X}, t_k) = 0, \quad \text{and with}$$

$$\text{probability } 1 - \beta_r(w^k, \frac{c^k}{w^k}, \frac{d^k}{w^k}) / \beta_r(v^k, \frac{e^k}{v^k}, \frac{f^k}{v^k}) \quad \text{set } \hat{X}_r(t_k) = 1,$$

$$\bar{W}(\hat{X}, t_k) = \bar{w}^k \quad \text{and} \quad B_r(\hat{X}, t_k) = d_r^k.$$

Intuitively, under (b.2) one sees that component  $r$  has completed repair at time  $t_k$  in the realization of  $\hat{Y}$ . As for  $\hat{X}$ , component  $r$  either completed repair or not. In either case it is seen that (2.7) - (2.9) hold.

(b.3). If  $r \in \bar{w}^k \cap v^k$  then change the state of component  $r$  in either

$\hat{X}$  or  $\hat{Y}$  as follows:

(b.3.1). With probability  $\beta_r(w^k, \frac{c}{w}^k, \frac{d}{w}^k) / \eta_r(w^k, v^k, \frac{c}{w}^k, \frac{d}{w}^k, \frac{e}{v}^k, \frac{f}{v}^k)$  set

$$\begin{aligned} \hat{X}_r(t_k) &= 0, \quad \bar{W}(\hat{X}, t_k) = \bar{w}^k - \{r\}, \quad A_r(\hat{X}, t_k) = 0, \\ \hat{Y}_r(t_k) &= \hat{Y}_r(t_k^-) = 0, \quad W(\hat{Y}, t_k) = v^k \quad \text{and} \quad A_r(\hat{Y}, t_k) = e_r^k. \end{aligned}$$

(b.3.2). With probability  $\gamma_r(v^k, \frac{e}{v}^k, \frac{f}{v}^k) / \eta_r(w^k, v^k, \frac{c}{w}^k, \frac{d}{w}^k, \frac{e}{v}^k, \frac{f}{v}^k)$  set

$$\begin{aligned} \hat{Y}_r(t_k) &= 1, \quad W(\hat{Y}, t_k) = v^k - \{r\}, \quad B_r(\hat{Y}, t_k) = 0, \\ \hat{X}_r(t_k) &= \hat{X}_r(t_k^-) = 1, \quad W(\hat{X}, t_k) = w^k \quad \text{and} \quad B_r(\hat{X}, t_k) = d_r^k. \end{aligned}$$

Intuitively it is seen that under (b.3.1), component  $r$  completes repair at time  $t_k$  in the realization of  $\hat{X}$ . However, in the realization of  $\hat{Y}$  component  $r$  is working anyway at time  $t_k$ . Under (b.3.2) component  $r$  fails at time  $t_k$  in the realization of  $\hat{Y}$ . However, in the realization of  $\hat{X}$  component  $r$  is under repair anyway at time  $t_k$ . In either case it is seen that (2.7) - (2.9) hold.

Let  $t_{k+1}$  be the time of the  $(k+1)$ -st epoch of  $\sigma$ . Now define

$\hat{X}$  and  $\hat{Y}$  on  $[t_k, t_{k+1})$  as follows: For  $t \in [t_k, t_{k+1})$  let  $\hat{X}(t) = \hat{X}(t_k)$  and  $\hat{Y}(t) = \hat{Y}(t_k)$ . Thus

$$(2.10) \quad \hat{X}(t) = \hat{Y}(t), \quad t \in [t_k, t_{k+1}).$$

Also for  $t \in [t_k, t_{k+1})$  let  $W(\hat{X}, t) = W(\hat{X}, t_k) = w^{k+1}$ ,

$$W(\hat{Y}, t) = W(\hat{Y}, t_k) = v^{k+1}, \quad A_w(\hat{X}, t)(\hat{X}, t) = A_w(\hat{X}, t_k)(\hat{X}, t_k) + (t - t_k)1,$$

$$B_w(\hat{X}, t)(\hat{X}, t) = B_w(\hat{X}, t_k)(\hat{X}, t_k) + (t - t_k)1,$$

$A_W(\hat{Y}, t)(\hat{Y}, t) = A_W(\hat{Y}, t_k)(\hat{Y}, t_k) + (t - t_k)1$ ,  $B_W(\hat{Y}, t)(\hat{Y}, t) =$   
 $B_W(\hat{Y}, t_k)(\hat{Y}, t_k) + (t - t_k)1$ . Thus, for  $t \in [t_k, t_{k+1})$ ,  
 $A_W(\hat{X}, t)(\hat{X}, t) \leq A_W(\hat{X}, t)(\hat{Y}, t)$  and  $B_W(\hat{Y}, t)(\hat{X}, t) \geq B_W(\hat{Y}, t)(\hat{Y}, t)$ . In  
 particular, denoting the realizations at time  $t_{k+1}^-$  by

$$\begin{aligned}
 \frac{c}{w}^{k+1} &\equiv A_W(\hat{X}, t_{k+1}^-)(\hat{X}, t_{k+1}^-), \\
 \frac{d}{w}^{k+1} &\equiv B_W(\hat{X}, t_{k+1}^-)(\hat{X}, t_{k+1}^-), \quad \frac{e}{v}^{k+1} \equiv A_W(\hat{Y}, t_{k+1}^-)(\hat{Y}, t_{k+1}^-), \\
 \frac{f}{w}^{k+1} &\equiv B_W(\hat{Y}, t_{k+1}^-)(\hat{Y}, t_{k+1}^-), \quad \text{it is seen that } \frac{c}{w}^{k+1} \leq \frac{e}{w}^{k+1} \quad \text{and} \\
 \frac{d}{v}^{k+1} &\geq \frac{f}{v}^{k+1}.
 \end{aligned}$$

Thus, by induction, the procedure described above defines

$\hat{X}(t)$  and  $\hat{Y}(t)$  for all  $t \geq 0$ .

From (2.6) and (2.10) it is seen that  $\hat{X}(t) \geq \hat{Y}(t)$ ,  $t \geq 0$ , with probability one. Using well known results on thinning of Poisson processes it is not hard to verify that  $\hat{X} \stackrel{st}{\leq} \underline{X}$  and  $\hat{Y} \stackrel{st}{\leq} \underline{Y}$ . The desired result then follows from (2.1).

Remark 2.3. In some applications of Theorem 2.2 (see, for example, Section 3) the  $\alpha_i$ 's,  $\beta_j$ 's,  $\gamma_i$ 's and  $\delta_j$ 's (see (2.3) and (2.4)) do not depend on the ages of the working components,  $\frac{c}{w}$  and  $\frac{e}{v}$ . The result of Theorem 2.2 then is still true even if (2.5.ii) does not hold. The proof of this statement is the same as the proof of Theorem 2.2. Similarly, if the  $\alpha_i$ 's,  $\beta_j$ 's,  $\gamma_i$ 's and  $\delta_j$ 's do not depend on  $\frac{d}{w}$  and  $\frac{f}{v}$ , then the conclusion of Theorem 2.2 is still true even (2.5.iii) does not hold.

Let  $\phi: \{0, 1\}^n \rightarrow \{0, 1\}$  be any coherent structure function of  $n$  components and let  $\underline{X} = \{X_1(t), \dots, X_n(t)\}$ ,  $t \geq 0$  and  $\underline{Y} = \{Y_1(t), \dots, Y_n(t)\}$ ,  $t \geq 0$  be the performance processes of the two DMRS's. The times of the first failure for each of the two DMRS's then are  $T_{\underline{X}} \equiv \inf\{t: \phi(\underline{X}(t)) = 1\}$  and

$T_Y \equiv \inf\{t: \phi(Y(t)) = 1\}$ , respectively.

Corollary 2.4. If the two DMRS's satisfy Assumption 2.1 and (2.3) and (2.4) and the initial states and ages of the up and down periods of the two DMRS's satisfy (2.5.i) - (2.5.iii), then

$$(2.11) \quad T_X \stackrel{st}{\leq} T_Y.$$

Proof. Let  $U = \{z \in \{0,1\}^n : \phi(z) = 1\}$  and let  $\hat{X}$  and  $\hat{Y}$  be as in the proof of Theorem 2.2. Then  $T_X \stackrel{st}{\leq} T_X^{\hat{}} \equiv \inf\{t: \hat{X}(t) \in U\}$ ,  $T_Y \stackrel{st}{\leq} T_Y^{\hat{}} \equiv \inf\{t: \hat{Y}(t) \in U\}$ . Let  $\underline{x}$  and  $\underline{y}$  be members of  $\{0,1\}^n$ . Note that if  $\underline{x} \notin U$  and  $\underline{y} < \underline{x}$  (i.e.,  $x_i < y_i$ ,  $i=1, \dots, n$ ) then  $\underline{y} \notin U$ . By Theorem 2.2,  $\hat{X}(t) > \hat{Y}(t)$  a.s. Hence  $T_X^{\hat{}} < T_Y^{\hat{}}$  a.s. and (2.11) follows.

In a similar manner, using Remark 2.3, one can obtain:

Corollary 2.5. Assume that the failure and repair completion rates do not depend on the ages of the working components. If the two DMRS's satisfy Assumption 2.1 and (2.3) and (2.4) and the initial states and ages of the up and down periods of the two DMRS's satisfy (2.5.i) and (2.5.iii), then

$$T_X \stackrel{st}{\leq} T_Y.$$

Remark 2.6. The assumption used in Theorem 2.2 and Corollaries 2.4 and 2.5 that the failure and repair completion rates depend on the up and down ages  $c_i$ ,  $i \in w$ ,  $d_j$ ,  $j \in \bar{w}$ , can clearly be modified to assuming that these rates depend on some increasing functions  $g_i(c_i)$ ,  $i \in w$ ,  $h_j(d_j)$ ,  $j \in \bar{w}$  of the ages. This modification shows that Theorem 2.2 and Corollaries 2.4 and 2.5 are useful in applications such as follows. Suppose that an "item" is a



rubber container which is being filled with some liquid at a constant rate. As the amount of liquid increases the container expands and becomes weaker and eventually "fails" by cracking. It is often the case, that the failure rate of the "item" (and of other "items") depend on the amount of liquid not linearly but through some nondecreasing function  $g$  of the amount of liquid. Theorem 2.2 and Corollaries 2.4 and 2.5 still apply in such cases.

### 3. NBU properties of some DMRS's.

In this section we consider only one DMRS at a time. We provide sufficient conditions on the component dynamics so that the system lifetime is NBU without any specific assumptions on the system structure  $\phi$ . For this single DMRS, the notation of Section 1 will be used. Throughout this section we assume:

Assumption 3.1. The failure and the repair completion rates do not depend on the ages of the working components.

We still allow the rates to depend on the ages of the repair times.

Thus, in this section, (1.1) and (1.2) reduce to

$$\begin{aligned}
 (3.1) \quad \lambda_k(w, \underline{b}_w) &\equiv \lim_{\Delta t \downarrow 0} \frac{1}{\Delta t} P\{Z_k(t+\Delta t) = 1 \mid W(\underline{Z}, t) = w, \underline{A}_w(\underline{Z}, t) = \underline{a}_w, \\
 &\quad \underline{B}_w(\underline{Z}, t) = \underline{b}_w^{-1}, k \in w, \\
 (3.2) \quad \mu_k(w, \underline{b}_w) &\equiv \lim_{\Delta t \downarrow 0} \frac{1}{\Delta t} P\{Z_k(t+\Delta t) = 0 \mid W(\underline{Z}, t) = w, \underline{A}_w(\underline{Z}, t) = \underline{a}_w, \\
 &\quad \underline{B}_w(\underline{Z}, t) = \underline{b}_w^{-1}, k \in \bar{w}.
 \end{aligned}$$

Since in this section the  $\lambda_k$ 's and  $\mu_k$ 's do not depend on  $\underline{a}_w$ , it follows that, given  $W(\underline{Z}, t) = w$ ,  $\underline{A}_w(\underline{Z}, t) = \underline{a}_w$  and  $\underline{B}_w(\underline{Z}, t) = \underline{b}_w^{-1}$ , the stochastic behaviour of  $\{Z(t), t \geq 0 \mid w, \underline{a}_w, \underline{b}_w\}$  does not depend on  $\underline{a}_w$ .

Thus, in this section, such a process will be denoted by  $\{Z(t), t \geq 0 | w, \underline{b}_w\}$  and when  $w = S \equiv \{1, \dots, n\}$ , we simply denote the process then by  $[Z|S]$  or by  $\{Z(t), t \geq 0 | S\}$ .

We will also need a counterpart of Assumption 2.1.

Assumption 3.2. For  $w \subset v \subset S$  and  $\underline{d}_w, \underline{f}_w$  denote

$$\begin{aligned} \xi_i(w, v, \underline{d}_w, \underline{f}_v) &= \lambda_i(w, \underline{d}_w) && \text{if } i \in w, \\ &= \mu_i(v, \underline{f}_v) && \text{if } i \in \bar{v}, \\ &= \lambda_i(w, \underline{d}_w) + \mu_i(v, \underline{f}_v) && \text{if } i \in \bar{w} \cap v. \end{aligned}$$

Assume that

$$\tilde{\lambda} \equiv \sup_{i=1}^n \xi_i(w, v, \underline{d}_w, \underline{f}_v) : w \subset v \subset S, \underline{d}_w > \underline{f}_v < \infty.$$

Theorem 3.3. Assume Assumptions 3.1 and 3.2 and that for all  $w \subset v \subset S$  and  $\underline{d}_w$  and  $\underline{f}_v$  such that  $\underline{d}_w > \underline{f}_v$ ,

$$(3.3) \quad \lambda_i(w, \underline{d}_w) > \lambda_i(v, \underline{f}_v) \quad , \quad i \in w \subset v,$$

$$(3.4) \quad \mu_i(w, \underline{d}_w) < \mu_i(v, \underline{f}_v) \quad \text{if } i \in \bar{v} \subset \bar{w}.$$

Then, for all choices of  $w^1$  and  $\frac{b^1}{w^1}$ ,

$$(3.5) \quad \{Z(t), t \geq 0 | w^1, \frac{b^1}{w^1}\} \stackrel{st}{=} \{Z(t), t \geq 0 | S\}.$$

Proof. We will use Theorem 2.2. Denote by  $\{Y(t), t \geq 0\}$  the right hand side of (3.5) and by  $\{X(t), t \geq 0\}$  the left hand side of (3.5). Then the

$\alpha_i$ 's,  $\beta_j$ 's,  $\gamma_i$ 's and  $\delta_j$ 's in (2.3) and (2.4) are identified as

$$\begin{aligned}\alpha_i(w, \underline{c}_w, \underline{d}_w) &= \lambda_i(w, \underline{d}_w), & i \in w, \\ \beta_j(w, \underline{c}_w, \underline{d}_w) &= \mu_j(w, \underline{d}_w), & j \in \bar{w}, \\ \gamma_i(v, \underline{e}_v, \underline{f}_v) &= \lambda_i(v, \underline{f}_v), & i \in v, \\ \delta_j(v, \underline{e}_v, \underline{f}_v) &= \mu_j(v, \underline{f}_v), & j \in \bar{v}.\end{aligned}$$

Thus, from (3.3) and (3.4) it follows that (2.3) and (2.4) hold.

Assumption 3.2 ensures that (2.2) and hence Assumption 2.1 hold.

Finally, since  $\underline{Y}(0) = \underline{0}$  a.s., it follows that (2.5.i) and (2.5.iii) hold.

By Remark 2.3, then,  $\underline{X} \stackrel{st}{\geq} \underline{Y}$ .

Consider a DMRS which starts to function with all components being up. That is  $\underline{Z}(0) = \underline{0}$ . Let  $T = \inf\{t: \dot{\mathbb{P}}(\underline{Z}(t)) = 1\}$  be the time until first system failure.

**Theorem 3.4.** For an  $n$ -component DMRS assume that the failure and repair completion rates satisfy the conditions of Theorem 3.3. If the system starts to function with all components being up then  $T$  is NBU.

Proof. Fix a  $t > 0$  and consider some particular realization

$\{\underline{z}(s): 0 \leq s \leq t\}$  of  $\{\underline{Z}(s): 0 \leq s \leq t\}$  such that  $\dot{\mathbb{P}}(\underline{z}(s)) = 0$ ,  $s \leq t$ , that is, given  $H_t \equiv \{\underline{Z}(s) = \underline{z}(s), s \leq t\}$  we have  $P(T > t | H_t) = 1$ . It will be argued below that for every such history  $H_t$ ,

$$(3.6) \quad [(T-t) | H_t] \stackrel{st}{\geq} T.$$

It follows then, by unconditioning in (3.6) but retaining the condition  $T >$

$t$ , that  $[(T-t)|T>t] \stackrel{st}{\leq} T$ , that is,  $T$  is NBU.

The stochastic ordering (3.6) follows from Theorem 3.3 in the same manner that Corollary 2.4 (and Corollary 2.5) follow from Theorem 2.2.

Remark. As noted in Remark 2.6, the failure and repair completion rates need not depend directly on the down ages for the conclusion of Theorem 3.4 to be true. This conclusion is also true when the rates depend on the ages through some increasing functions.

Motivated by Barlow and Proschan (1976) and by Chiang and Niu (1980) we state a slight generalization of Theorem 3.4. We assume now that the DMRS has  $n$  (dependent) repairable components, as described in Section 1, and also  $m$  nonrepairable components with lifetimes which are independent of each other and of the states and the current ages and repair durations of the repairable components.

Theorem 3.5. For a DMRS with  $n + m$  components as described above assume:

- (a) the nonrepairable components have NBU lifetimes,
- (b) the failure and repair completion rates of the repairable components satisfy the conditions of Theorem 3.3,
- (c) all components are new at time 0.

Then the time until first system failure is NBU.

The proof of Theorem 3.5 is similar to the proof of Theorems 3.3 and 3.4. The additional ingredient is that  $H_t$  now also contains the information whether and which of the nonrepairable component are alive or dead at time  $t$ . The nonrepairable components which are alive (according to the history  $H_t$ ) at time  $t$  have stochastically smaller residual lives than the same new components (here we use the assumption that their lifetimes are NBU).

Defining on the same probability space the processes

$\hat{X} = \{(\hat{X}_1(t), \dots, \hat{X}_n(t), \hat{X}_{n+1}(t), \dots, \hat{X}_{n+m}(t)), t \geq 0\}$  [the last  $m$  coordinates of  $\hat{X}$  correspond to the nonrepairable components] and  $\hat{Y} = \{(\hat{Y}_1(t), \dots, \hat{Y}_{n+m}(t)), t \geq 0\}$  such that  $\hat{Y} \stackrel{st}{=} [Z|S]$  and  $\hat{X} \stackrel{st}{=} [Z(t+)|H_t]$ , in a manner similar to the proof of Theorem 2.2, it is seen that  $\hat{X} \geq \hat{Y}$  a.s. The result then follows by partially unconditioning as in the proof of Theorem 3.4. We omit the details.

The conclusion of Theorem 3.5 still holds even if (a) is relaxed as follows. Assume as before that the lifetimes of the nonrepairable components are independent of the states and the current ages and repair durations of the repairable components. However, allow now the joint distribution of the lifetimes of the nonrepairable components to be dependent in the sense of Arjas (1981), that is, assume that

(a') the nonrepairable components have MNBU  $|\mathcal{F}_t$  joint distribution with  $\mathcal{F}_t$  being the  $\sigma$ -field generated by the nonrepairable components.

The following result (whose proof is similar to the proof of Theorem 3.5) is valid:

Theorem 3.6. For a DMRS with  $n + m$  components as described above assume (a'), (b) and (c). Then the time until first system failure is NEU.

When the repairable components evolve independently of each other, that is,  $\lambda_i(w, \underline{d}_w) = \lambda_i$ ,  $\mu_i(w, \underline{d}_w) = \mu_i(d_i)$ , then Theorem 3.5 (or 3.6) reduces to the results of Barlow and Proschan (1976), Theorem 2.7 and Chiang and Niu (1980), Theorem 3.5. In this context a special case of Theorem 3.3 is for a single component subject to failures and repairs. Suppose that the single component stays up (in state 0) for an exponential time period and stays down (in state 1) for a DFR time period. Then the component performance process

$Z$  satisfies the stochastic monotonicity

$$(3.7) \quad \{Z|Z(0) = 1, B_1(Z, 0) = b\} \stackrel{st}{\geq} \{Z|Z(0) = 0\} \quad \text{for all } b > 0$$

(see, e.g., Barlow and Proschan (1976), Lemma 2.5, Chiang and Niu (1980), Lemma 3.1 and Shanthikumar (1984), Theorem 5.1). Miller (1979) has generalized this result and showed that if

(d) the up-periods are exponential,

(e) the down-periods are NWU,

then (3.7) holds. The NWU condition is weaker than the DFR condition.

Miller's result thus tempts one to weaken (3.4) to a condition analogous to NWU. However, in the next subsection it is shown through a counterexample that such a modification is not possible.

### 3.1. A counterexample.

In this subsection we construct a class of univariate performance processes  $Z = \{Z(t), t \geq 0\}$  which satisfy conditions (d) and (e) but violates (3.7).

For a fixed  $\varepsilon > 0$ ,  $\tau > 0$  and  $\mu > 0$  let  $V$  be a random variable with survival function

$$\begin{aligned} \bar{G}(v) &= e^{-\mu(v-r\varepsilon)} \quad \text{if } v \in [(\tau+\varepsilon)r, (\tau+\varepsilon)r+\varepsilon], \quad r = 0, 1, 2, \dots, \\ &= e^{-\mu(r+1)\varepsilon} \quad \text{if } v \in [(\tau+\varepsilon)r+\varepsilon, (\tau+\varepsilon)(r+1)], \quad r = 0, 1, 2, \dots \end{aligned}$$

A graph of  $-\log \bar{G}(v)$  is shown in Figure 3.1. It is easy to verify that  $\bar{G}$  is NWU.

Let  $Z = \{Z(t), t \geq 0\}$  be a component performance process that has

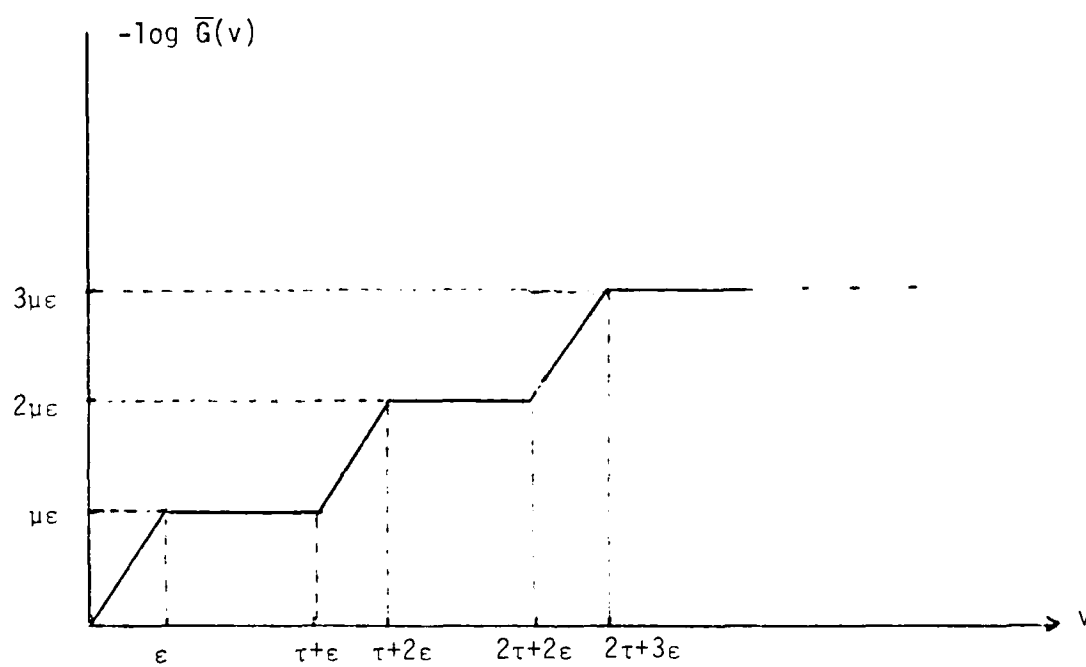


Figure 3.1

exponentially distributed up period with mean  $\lambda^{-1}$  (which will be a fixed constant throughout this subsection) and down periods with survival function  $\bar{G}$ .

Let  $\Delta \in (0, \min(\varepsilon, \tau))$  and define

$$\begin{aligned} \{Z^{(1)}(t), t \geq 0\} &\stackrel{\text{st}}{=} \{Z(t), t \geq 0 \mid Z(0) = 0\}, \\ \{Z^{(2)}(t), t \geq 0\} &\stackrel{\text{st}}{=} \{Z(t), t \geq 0 \mid Z(0) = 1, B_1(Z, 0) = \varepsilon + \Delta\}. \end{aligned}$$

That is,  $Z^{(2)}$  starts at state 1 at time 0 with an elapsed repair time of  $\varepsilon + \Delta$ . Since  $0 < \Delta < \tau$ , it is clear that no repair completion will take place for the next  $\tau - \Delta$  time units (since  $\bar{G}(t + \varepsilon + \Delta) / \bar{G}(\varepsilon + \Delta) = 1$ ,  $0 \leq t \leq \tau - \Delta$ ). Then, conditioning on the repair completion instance after time  $\tau - \Delta$  and no failures until time  $\tau$  (recalling that  $\Delta \leq \varepsilon$ ) one has

$$(3.8) \quad P\{Z^{(2)}(\tau) = 0\} = \int_{\alpha=0}^{\Delta} \mu e^{-\mu\alpha} e^{-\lambda(\Delta-\alpha)} d\alpha \equiv h_{\Delta, \lambda}(\mu),$$

where

$$\begin{aligned} h_{\Delta, \lambda}(\mu) &= \frac{\mu}{\mu - \lambda} (e^{-\lambda\Delta} - e^{-\mu\Delta}) \quad \text{if } \mu \neq \lambda, \\ &= \Delta e^{-\lambda\Delta} \quad \text{if } \mu = \lambda. \end{aligned}$$

Note that  $h_{\Delta, \lambda}(\mu)$  is a continuous function of  $\mu$  and that

$$\begin{aligned} h_{\Delta, \lambda}(\mu) &\rightarrow e^{-\lambda\Delta} \quad \text{as } \mu \rightarrow \infty, \\ &\rightarrow 0 \quad \text{as } \mu \rightarrow 0. \end{aligned}$$

Thus



$$(*) \quad \begin{cases} \text{given } \varepsilon > 0, \Delta \varepsilon(0, \varepsilon) \text{ and } \varepsilon_0 < e^{-\Delta \lambda}, \\ \text{there exists a } \mu = \mu_{\varepsilon_0, \Delta} \text{ such that,} \\ \text{for all } \tau > \Delta, \text{ we have} \\ P\{Z^{(2)}(\tau) = 0\} > h_{\Delta, \lambda}(\mu) > \varepsilon_0. \end{cases}$$

Now consider the performance process  $Z^{(1)}$  and its associated age process  $\{B_1(Z^{(1)}, t), t \geq 0\}$ . When  $Z^{(1)}(t) = 0$  then  $B_1(Z^{(1)}, t)$  is vacuous. For technical reasons it is convenient to define then  $B_1(Z^{(1)}, t) \equiv 0$  so that  $B_1(Z^{(1)}, t)$  is well defined for all  $t \geq 0$ . Let  $q(x)$  be the probability that  $Z^{(1)}$  has not been in state 1 continuously for more than  $\varepsilon$  time units during  $[0, x]$ , i.e.,

$$q(x) \equiv P\left\{\sup_{0 \leq t \leq x} B_1(Z^{(1)}, t) \leq \varepsilon\right\}, \quad x \geq 0.$$

Since  $\bar{G}(\tau + \varepsilon)/\bar{G}(\varepsilon) = 1$ , it is clear that

$$\left\{\sup_{0 \leq t \leq x} B_1(Z^{(1)}, t) > \varepsilon\right\} \subset \{Z^{(1)}(x) = 1\}, \quad 0 \leq x \leq \tau + \varepsilon.$$

Therefore

$$P\{Z^{(1)}(x) = 0\} \leq q(x), \quad 0 \leq x \leq \tau + \varepsilon.$$

Note that for all  $x \geq 0$ ,  $q(x)$  does not depend on  $\tau$ . In fact, conditioning on the first transition of  $Z^{(1)}$  from state 0 to 1, one has

$$(3.9) \quad q(x) = \int_{z=0}^{x-\varepsilon} \lambda e^{-\lambda z} \int_{u=0}^{\varepsilon} \mu e^{-\mu u} q(x-z-u) du dz + e^{-\lambda(x-\varepsilon)}, \quad x > \varepsilon,$$

and

$$q(x) = 1, \quad 0 \leq x \leq \varepsilon.$$

Taking the Laplace transforms on both sides of (3.9) and solving for

$$\hat{q}(s) = \int_0^\infty e^{-sx} q(x) dx, \quad \text{one can show that}$$

$$(3.10) \quad \lim_{x \rightarrow \infty} q(x) = \lim_{s \rightarrow 0} s \hat{q}(s) = 0.$$

Thus

$$(**) \quad \begin{cases} \text{given } \varepsilon > 0, \mu > 0, \varepsilon_0 > 0 \text{ there exists} \\ \text{a } \tau = \tau_{\varepsilon, \mu, \varepsilon_0} \text{ such that} \\ P\{Z^{(1)}(\tau) = 0\} \leq q(\tau) < \varepsilon_0. \end{cases}$$

Now, fix  $\varepsilon > 0$ ,  $\Delta \in (0, \varepsilon)$ ,  $\varepsilon_0 \in (0, e^{-\Delta\lambda}]$ . Determine  $\mu = \mu_{\varepsilon_0, \Delta}$  as in (\*). Determine  $\tau = \tau_{\varepsilon, \mu_{\varepsilon_0, \Delta}, \varepsilon_0}$  as in (\*\*). [If this  $\tau$  is not  $> \Delta$  then choose a bigger  $\tau > 0$  which satisfies (\*\*). Such a  $\tau$  exists by (3.10).] Then, for these  $\mu$  and  $\tau$ ,

$$P\{Z^{(2)}(\tau) = 0\} \underset{(*)}{>} \varepsilon_0 \underset{(**)}{>} P\{Z^{(1)}(\tau) = 0\}$$

in contradiction to (3.7) and to Theorem 3 of Miller (1979).

#### 4. Discussion and further results.

It is fair to say that the set of results obtained in this paper using the method of "putting two processes on the same probability space" is only a sample of the kind of results which can be obtained by this method. Some further results and applications are indicated below.

4.1. NBU-ness of some first passage times. One can generalize Theorem 3.3 as follows: Let  $\mathcal{A} \equiv 2^S \times [0, \infty)^n$  where  $2^S$  is the set of all subsets of

$S = \{1, 2, \dots, n\}$ . Denote a generic element of  $\mathcal{A}$  by  $(w, \underline{b})$ ,  $w \subset S$ ,  $\underline{b} \geq \underline{0}$ . Denote by  $\tilde{\underline{b}}_{\bar{w}}$  the vector  $(b_1, \dots, b_n)$  where for each  $i \in w$ ,  $b_i$  is replaced by 0. Consider the state space  $\mathcal{B} \equiv \{(w, \tilde{\underline{b}}_{\bar{w}}), w \subset S, \underline{b} \geq \underline{0}\} \subset \mathcal{A}$ . Define a partial ordering on  $\mathcal{B}$  by  $(w, \tilde{\underline{b}}_{\bar{w}}) \leq (v, \tilde{\underline{b}}'_{\bar{v}})$  if and only if  $\bar{w} \subset \bar{v}$  and  $\tilde{\underline{b}}_{\bar{w}} \leq \tilde{\underline{b}}'_{\bar{v}}$ . Consider the right continuous process

$$(4.1) \quad M = \{M(t), t \geq 0\} \equiv \{(W(\underline{Z}, t), \tilde{\underline{B}}_{\bar{W}}(\underline{Z}, t)(\underline{Z}, t)), t \geq 0\}$$

on  $\mathcal{B}$ , where  $W(\underline{Z}, t)$  is as defined in Section 1 and  $\tilde{\underline{B}}_{\bar{W}}(\underline{Z}, t)$  is the vector with  $i$ -th coordinate being  $B_i(\underline{Z}, t)$  [as defined in Section 1] if  $i \in \bar{w}$  and the  $j$ -th coordinate being 0 if  $j \in w$ . Note that if  $\underline{Z}$  is a performance process of a DMRS which satisfies Assumption 3.2, then  $M$  is a Markov process on  $\mathcal{B}$ .

Using a proof similar to the proof of Theorem 3.3, it can be shown, under Assumptions 3.1, 3.2 and conditions (3.3) and (3.4), that  $M$  is stochastically monotone with respect to the ordering  $\leq$  on  $\mathcal{B}$ , that is,

$$(4.2) \quad \{M(t), t \geq 0 \mid M(0) = (w, \tilde{\underline{b}}_{\bar{w}})\} \stackrel{st}{\sim} \{M(t), t \geq 0 \mid M(0) = (v, \tilde{\underline{b}}'_{\bar{v}})\} \text{ if } \bar{w} \subset \bar{v}, \tilde{\underline{b}}_{\bar{w}} \leq \tilde{\underline{b}}'_{\bar{v}},$$

where the notation  $\stackrel{st}{\sim}$  is self-explanatory. Note that (4.2) is a special case of (4.2). A straightforward extension of Lemma 2.3 of Barlow and Proschan (1976) yields the following (consult Marshall and Shaked (1985) and Shaked and Shanthikumar (1984) for the measure-theoretic considerations). If  $M(0) = (S, \underline{0})$  a.s. then  $M$  is an NBU process on  $\mathcal{B}$ , that is, for every closed upper set  $U \subset \mathcal{B}$  (a set  $U$  is an upper set if  $(v, \tilde{\underline{b}}'_{\bar{v}}) \in U$  whenever  $(v, \tilde{\underline{b}}'_{\bar{v}}) \geq (w, \tilde{\underline{b}}_{\bar{w}})$  for some  $(w, \tilde{\underline{b}}_{\bar{w}}) \in U$ ), the first passage time

$$(4.3) \quad T_U = \inf\{t: N(t) \in U\} \quad \text{is NBU}$$

(see Marshall and Shaked (1985) for a discussion on NBU processes with general state space).

4.2. Multistate coherent structures. Let  $\Phi(z_1, \dots, z_n)$  be a multistate coherent structure function, i.e.,  $\Phi(\underline{z})$  is a nondecreasing function of  $\underline{z} \in \{0, 1\}^n$  (with a real range which may be more general than  $\{0, 1\}$ ). If Assumptions 3.1 and 3.2 hold and if conditions (3.3) and (3.4) are satisfied and if  $\underline{Z}(0) = \underline{0}$  a.s., then a special case of (4.3) is

$$T_u = \inf\{t: \Phi(\underline{Z}(t)) \geq u\} \quad \text{is NBU, } u \geq 0.$$

4.3. Total ages of items under repair. From Theorem 3.8 of Marshall and Shaked (1985) it follows that for every nondecreasing continuous function  $\psi: \mathbb{R}_+^n \rightarrow \mathbb{R}$ , such that  $\psi(\underline{0}) = 0$ , the random time

$$(4.4) \quad S_u = \inf\{t: \tilde{R}_W^-(\underline{Z}, t)(\underline{Z}, t) \geq u\} \quad \text{is NBU, } u \geq 0,$$

where here  $\tilde{R}_W^-(\underline{Z}, t)$  is as defined in (4.11). For example, from (4.4) it follows that if a DMRS satisfies Assumptions 3.1 and 3.2 and conditions (3.3) and (3.4) and has all components new at time 0, then the first time at which the total ages of items under repair exceeds  $u$ , is NBU.

4.4. A DMRS with a backup unit. Consider a DMRS as described in Section 1 and suppose that a backup unit with a random life  $L$  is available. The backup unit works only during the down periods of the system (provided it has

not already failed). During the up periods the backup unit is automatically switched off. Denote by  $D(t) \equiv \int_0^t \phi(\underline{Z}(u))du$  the cumulative down time of the DMRS where here  $\phi$  is a binary nondecreasing function. Backed by the backup unit, the DMRS experiences its first actual failure at the random time

$$T_L \equiv \inf\{t \geq 0: D(t) \geq L\}.$$

We will show now that if Assumptions 3.1 and 3.2 and conditions (3.3) and (3.4) hold and if  $L$  is NBU and independent of  $\{M(t), t \geq 0\}$  of (4.1), then  $T_L$  is NBU.

Since  $D$  is an increasing functional of  $M$ , it follows that for every  $w \in S$ ,  $\underline{b}_w$  and  $s \geq 0$ , we have

$$(4.5) \quad \{D(t), t \geq 0 | S, 0\} \stackrel{st}{\leq} \{D(s+t), t \geq 0 | \underline{Z}(u) = \underline{z}(u), 0 \leq u \leq s, W(\underline{Z}, s) = w, \underline{B}_w(\underline{Z}, s) = \underline{b}_w^{-1}\}.$$

Since  $L$  is NBU, we have

$$(4.6) \quad L \stackrel{st}{\geq} [L-s | L > s], \quad s \geq 0.$$

Conditioning on the history  $\underline{Z}(u) = \underline{z}(u)$ ,  $0 \leq u \leq s$ ,  $W(\underline{Z}, s) = w$ ,

$\underline{B}_w(\underline{Z}, s) = \underline{b}_w^{-1}$  of  $M$  up to time  $s$  and on  $\{L > \int_0^s \phi(\underline{z}(u))du\}$ , we obtain from (4.5), (4.6) and the independence of  $M$  and  $L$ , that for  $s \geq 0$ ,

$$\begin{aligned} \{D(t) - L, t \geq 0 | S, 0\} &\stackrel{st}{\leq} \{D(s+t) - L - d(s), t \geq 0 \\ &| \underline{Z}(u) = \underline{z}(u), 0 \leq u \leq s, W(\underline{Z}, s) = w, \underline{B}_w(\underline{Z}, s) = \underline{b}_w^{-1}, L > \int_0^s \phi(\underline{z}(u))du\}. \end{aligned}$$

where  $d(s) \equiv \int_0^s \phi(\underline{z}(u))du$ . Hence for  $s \neq 0$ ,

$$(4.7) \quad T_L \overset{st}{\geq} [\inf\{t \geq 0: D(s+t) - L - d(s) \geq 0 \mid \underline{Z}(u) = \underline{z}(u), 0 \leq u \leq s, \\ W(\underline{Z}, s) = w, \frac{B-(\underline{Z}, s)}{w} = \frac{b-w}{w}, L > d(s)\}].$$

Unconditioning in (4.7), retaining the condition  $T_L > s$ , we obtain

$$T_L \overset{st}{\geq} [T_L - s \mid T_L > s], \quad s \geq 0, \quad \text{that is, } T_L \text{ is NBU.}$$

4.5. Stochastic comparisons of up periods. We present a result that compares the first up period (denoted by  $V_1$ ) of a DMRS with its subsequent up periods  $V_2, V_3, \dots$ . Denote by  $U_1, U_2, \dots$  the sequence of down periods. Fix a positive integer  $m$ . Conditioning on the state of the process  $M$  at the random time  $\sum_{i=1}^{m-1} (V_i + U_i) + u, u \geq 0$ , under the condition  $\{V_m > u\}$ , one easily obtains

Theorem 4.1. Under the conditions of Theorem 3.3,

$$(4.8) \quad V_1 \overset{st}{\geq} [V_m - u \mid V_m > u], \quad u \geq 0, \quad m = 1, 2, \dots$$

The above theorem is a generalization of Theorem 3.6 of Chiang and Niu (1980). Such a generalization is needed in Shanthikumar (1985) to establish the NBU property of the first failure time of a dependent parallel system with safety periods.

Condition (4.8) is also sufficient for the validity of some results of Shanthikumar (1984). For example, Theorem 4.4 of Shanthikumar (1984) shows that the first passage time to overflow, in a dam with level dependent release rule and compound renewal input, is NBU whenever the inter-renewal times of the input satisfy (4.8).

4.6. DMRS's with safety periods. Consider a DMRS as described in Section 1. Suppose that every time the system is down, a random safety period is provided. Denote the  $i$ -th safety period (at the  $i$ -th time the system switches down) by  $Y_i$ ,  $i=1, 2, \dots$ . If the system is repaired before the safety period is over, then the system continues to operate normally. Otherwise, the operation stops. Let  $\tau$  be the first time in which the operation stops.

Result 4.2. Under the conditions of Theorem 3.3, if  $Y_1, Y_2, \dots$  are NBU, independent and identically distributed then  $\tau$  is NBU.

This result extends Corollary 5 of Shanthikumar (1985). The proof uses Theorem 4.1 and its idea is similar to the proof of the result in subsection 4.4. We omit the details.

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